Generalizing Hartogs’ Trichotomy Theorem

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Introduction

The Trichotomy Principle says that a pair of sets $A$ and $B$ either admits a bijection or else precisely one of these sets injects into the other. Hartogs established logical equivalence between the Trichotomy Principle and the Well-Ordering Principle. As ZF suffices to prove the Schröder-Bernstein theorem, the heart of Trichotomy lies in the existence of some injection connecting $A$ and $B$ (in either direction).

Hartogs' theorem seems a remarkable achievement, even judged against the large industry that flowered on the theme of statements logically equivalent to the Axiom of Choice. Certainly Trichotomy follows quickly from the Well-Ordering Principle; but deriving enough structure out of Trichotomy to well-order an arbitrary set might seem daunting.

Hartogs had the remarkable idea of associating to any set $A$ a certain ordinal that we write here as $\omega(A)$. To recall the definition of $\omega(A)$, we adopt von Neumann’s formulation of ordinals. Von Neumann views ordinals as hereditarily transitive sets; transitive means that elements constitute subsets and hereditarily transitive means transitive with only transitive elements. An ordinal, viewed as an ordered set, contains as elements just some other ordinals ordered by inclusion (or equivalently, by membership); in fact an ordinal’s elements coincide with its proper initial segments. Given any well-ordered set, one may produce (by transfinite induction) a unique ordinal with the same order type.

Hartogs first associates to a set $A$ the set $W(A)$ of all well-ordered sets modeled on subsets of $A$ and then defines the ordinal $\omega(A)$ as the set of all ordinals order-isomorphic to elements of $W(A)$. Hartogs argues that $\omega(A)$ cannot inject into $A$ (lest $\omega(A)$ contain itself). Then Trichotomy guarantees that $A$ injects into $\omega(A)$.

In this note we strengthen Hartogs’ theorem, by deriving the well-ordering principle from seemingly weaker statements.

We say a family of sets $\mathcal{F}$ contains an injective if there exists at least one
injective map \( i \) which has, for its source and target, distinct sets in \( \mathcal{F} \).

We say the \( k \)-Trichotomy Principle holds if every family of cardinality \( k \)
contains an injective; so 2-Trichotomy recaptures classical Trichotomy.

In this note we shall prove the

**Theorem:** For all finite \( k \), the \( k \)-Trichotomy Principle implies the Well-Ordering Principle.

**Notation and basic terminology**

For sets \( A \) and \( B \) we write \( A \leq B \) if some map injects \( A \) into \( B \). We write \( A \cong B \) if a bijection take \( A \) to \( B \). We write \( A < B \) if \( A \leq B \) but not \( A \cong B \).

We call a set *infinite* if every finite ordinal injects into it. We write \( \omega = \{0,1,2,\ldots\} \) for the smallest infinite ordinal. We view ordinals as sets of ordinals. *Cardinal* means an ordinal larger than any of its element.

We write \( A + B \) for the disjoint union of \( A \) and \( B \) and \( nA \) for the disjoint union of \( n \) copies of \( A \).

By a subquotient of a set \( A \), we mean any quotient set of a subset of \( A \).

We write \( 2^A \) for the powerset of \( A \).

**Proof of the Theorem**

In the sequel, a \((k)\) indicates a proposition whose validity depends on \( k \)-Trichotomy.

**Definition:** Given any set \( A \), well-orderable or not, we shall write \( \omega(A) \) for
the smallest ordinal, that does not inject into \( A \) (as per the Introduction).
\( \omega(A) \) always exists; it contains precisely the ordinals of all possible well-orderings of subsets of \( A \). For well-orderable \( A \), \( A < \omega(A) \); for non-well-orderable \( A \), no injective will connect \( A \) and \( \omega(A) \) in either direction.

By its definition, \( \omega(A) \) gives a strict upper bound on the cardinality of a
well-ordered set \( \kappa \) injecting *into* \( A \). But observe also that an injection from
\( A \) into \( B + \kappa \) hits fewer than \( \omega(A) \) elements of \( \kappa \).

\((k)\) **Lemma 1:** Fix a cardinal \( \kappa \). Given sets \( A_1 \leq \cdots \leq A_i \leq \cdots A_k \) with \( \omega(A_i) = \kappa \) for all \( A_i \), there exists \( n < m \) and a finite set \( R \) such that
\( A_m \leq A_n + R \); in case \( \kappa > \omega \), we even get \( A_m \cong A_n \).
Proof Fix a decreasing sequence of infinite cardinals, $\kappa_1, \ldots, \kappa_k = \omega(A_i)$. (For example, set $\kappa_j = \omega(\kappa_{j+1})$.)

Apply $k$-Trichotomy to the family $A_1 + \kappa_1, A_2 + \kappa_2, \ldots, A_k + \kappa_k$ to get an injection $A_m + \kappa_m \to A_n + \kappa_n$.

We must have $n < m$. Indeed, consider the restricted injection $\kappa_m \to A_n + \kappa_n$. Either the preimage of $A_n$ bijects with $\kappa_m$ (absurd since $\omega(A_n) = \kappa \leq \kappa_m$) or the preimage of $\kappa_n$ bijects with $\kappa_m$ (absurd since $\kappa_n < \kappa_m$).

Restrict $A_m + \kappa_m \to A_n + \kappa_n$ to $j : A_m \to A_n + \kappa_n$. We get $j^{-1}(\kappa_n) < \kappa$ (because $\omega(A_m) = \kappa$). Now write $R = j(j^{-1}(\kappa_n))$. Then $R < \kappa$, so $R$ injects into $A_n$ (since $\omega(A_n) = \kappa$ too).

If $\kappa = \omega$, we have $R$ finite.

If $\kappa > \omega$, $\omega$ also injects into $A_n$ (as $R$ does). So $A_n \cong S + R$, say, and also $A_n \cong T + \omega$, say. Now on $R$ infinite (and well-ordered) we have $R + R \cong R$, thus $A_n \cong S + R \cong S + R + R \cong A_n + R$; and on $R$ finite we have $A_n \cong T + \omega \cong T + \omega + R \cong A_n + R$. Either way, $A_n + R \cong A_n$. Thus $A_m$ injects into $A_n$. But $A_n \leq A_m$ by hypothesis, so Schroder-Bernstein finally yields $A_n \cong A_m$.♣

Lemma 2: For an infinite set $A$, the following are equivalent:

(i) There exists no injective map from $\omega$ to $A$;
(ii) $A$ bijects with no proper subset of $A$;
(iii) $\omega(A) = \omega$.

As usual, we call any such set $A$ infinite Dedekind finite.

Proof From (iii), which says all finite ordinals inject into $A$, but $\omega$ doesn’t, we get (i) (plus the infinitude of $A$); then (i) and the infinitude of $A$ amounts to (iii).

If (i) fails, we have an injection $w : \omega \to A$. To make (ii) fail, we manufacture a non-bijective injection $u : A \to A$. Set $u(w(k)) = w(k+1)$ for $k \in \omega$ and just have $u$ act as the identity on $A \setminus w(\omega)$. Clearly $A$ has the cardinality of $u(A) = A \setminus \{w(0)\}$, a proper subset.

If (ii) fails, we get a non-bijective injection $u : A \to A$. Fix $a \not\in u(A)$. We must have $a, u(a), u(u(a)), u(u(u(a))), \ldots, u^k(a), \ldots$ all distinct, so mapping $k$ to $u^k(a)$ makes (i) fail.♣
Dedekind finiteness for $A$ implies Dedekind finiteness for all $nA$, $n > 0$. (Whenever $\omega$ injects into $nA$, at least one copy of $A$ has an infinite preimage.)

(k) **Lemma 3**: No infinite Dedekind finite set exists.

**Proof** Apply Lemma 1 to $A \leq 2A \leq \cdots \leq kA$ to get an injection $mA \to nA + R$ with $m > n$ and $R < \omega(A)$.

We don’t have $A$ infinite Dedekind finite unless $\omega(A) = \omega$. But if so, that makes $R$ finite and then $A$ infinite makes $R < A$. Since $mA + R$ injects properly into $mA$, $mA$ injects properly into $mA$. Thus Dedekind finiteness fails for $mA$ (Lemma 2(ii)) and hence for $A$, contradiction. ♣

(k) **Lemma 4**: For every infinite set $A$ there exists $n$ such that $nA + nA \cong nA$.

**Proof** By Lemma 3 and Lemma 2(iii), $\omega(A) > \omega$. So apply Lemma 1 to $\{iA\}_{i=1,\ldots,k}$ to get $mA \cong nA$ with $m > n$.

Then $nA \cong mA \cong nA + (m - n)A \cong mA + (m - n)A \cong (2m - n)A$, and continuing this way, $nA \cong (m + q(m - n))$ for all $q$. Taking $q$ large, we may suppose we had $m > 2n$ in the first place. But $mA \cong nA$ certainly implies $(m + r)A \cong (n + r)A$, so we may even suppose we had $m = 2n$. ♣

**Note** In ZF, as Lindenbaum and Tarski show, $nP \cong nQ$ implies $P \cong Q$ (see Conway and Doyle).

**Lemma 5**: If a set $X$ admits a well-ordering, any subquotient $Y$ of $X$ does too.

**Proof** Fix a well-ordering of $X$. By definition, $Y$ bijects with a family $\{X_y\}_{y \in Y}$ of disjoint subsets of $X$. Each $X_y$ has a minimal element $m_y$ according to the order on $X$. By the disjointness of the $X_y$, $Y$ bijects with $\{m_y\}$, but $\{m_y\}$ has a well-ordering as a subset of $X$. ♣

**Lemma 6**: A set $A$ admits a well-ordering if $A \cong A + A$ and there exists an injection $k : 2^A \to A + \kappa$ for some ordinal $\kappa$.

**Proof** We aim to exhibit $A$ as a subquotient of $\kappa$ (Lemma 5).

Write $A = B + C$, with $A \cong B \cong C$; exhibiting $C$ as a subquotient of $\kappa$ suffices.

For $c \in C$, define $S_c := \{X \subseteq A | X \cap C = \{c\}\}$. Naturally $S_c \cong 2^B \cong 2^A$. Certainly $S_c \cap S_d = \emptyset$ unless $c = d$. Also $T_c := k(S_c) \cap \kappa \neq \emptyset$ since $k(S_c) \cong \ldots$
$2^A > A$ (Cantor). Therefore $\{T_e\}_{e \in C}$ forms a collection of pairwise disjoint nonempty sets in $\kappa$. ♣

**Proof of the Main Theorem:** Fix any set $S$. Set $A = nS$ with $n$ so large that $A + A \cong A$ (Lemma 4).

Write $\mathcal{P}^0(A) = A$, $\mathcal{P}^{i+1}(A) = 2^{\mathcal{P}^i(A)}$ and $\omega^0(A) = \omega(A)$, $\omega^{i+1}(A) = \omega(\omega^i(A))$. $k$-Trichotomy for the family $\{\mathcal{P}^i(A) + \omega^{k-i-1}(\mathcal{P}^{k-1}(A))\}_{i=0,\ldots,k-1}$ yields an injection

$$\mathcal{P}^i(A) + \omega^{k-i-1}(\mathcal{P}^{k-1}(A)) \to \mathcal{P}^j(A) + \omega^{k-j-1}(\mathcal{P}^{k-1}(A)).$$

Always $\omega^{k-i-1}(\mathcal{P}^{k-1}(A)) > \mathcal{P}^j(A)$; if $i < j$, also

$$\omega^{k-i-1}(\mathcal{P}^{k-1}(A)) > \omega^{k-j-1}(\mathcal{P}^{k-1}(A)).$$

So we must have $i > j$.

Our injection restricts to

$$\mathcal{P}^i(A) \to \mathcal{P}^j(A) + \omega^{k-j-1}(\mathcal{P}^{k-1}(A)),$$

so we get an injection

$$\mathcal{P}^i(A) \to \mathcal{P}^{i-1}(A) + \omega^{k-j-1}(\mathcal{P}^{k-1}(A)).$$

Thus $\mathcal{P}^{i-1}(A)$ satisfies the hypothesis of Lemma 6, so admits a well-ordering. But $A < 2^A$ (by singletons) and then $A < \mathcal{P}^{i-1}(A)$ (by induction), so a well-ordering of $\mathcal{P}^{i-1}(A)$ induces a well-ordering of $A$ which induces a well-ordering of $S$. ♣