Classical Mechanics: Inertia tensor activity

The purpose of this tutorial is to think about what it means for inertia to be tensor and not simply a scalar. We will look at this issue both mathematically and physically. Before doing this activity, you should have had some recent practice in finding eigenvalues and eigenvectors of 3×3 matrices, and you should have calculated the inertia tensor for one or two cases.

1. Let's look at the problem mathematically. In each of the following you are given an angular acceleration vector. In each case, find the torque using $\tau = I\alpha$ and using matrix multiplication. The inertia tensor for the block is given by

$$\{I\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

(a)

$$\vec{\alpha}_A = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(b)

$$\vec{\alpha}_B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(c)

$$\vec{\alpha}_C = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(d)

$$\vec{\alpha}_D = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

2. On this page we will connect the physical picture and the mathematical picture.

   (e) In the cases above, are τ and α always parallel or not? How can you tell?

   (f) State whether you agree or not with each of the students below. Give both mathematical and physical reasons for your point of view.

      (i) Student one: *The torque and angular acceleration* **have** *to be parallel, just like force and acceleration.*

      (ii) Student two: *No, they don't have to be parallel. I imagine a really, really long skinny bar. It seems clear in this case that torque and angular acceleration aren't parallel. It takes more effort to spin one way than another.*

(g) At this point, you might be a bit frustrated and confused. τ and α are not always parallel; this is very different from force and linear acceleration. On this page we will play with these ideas a bit more. Again, we present statements by other students. Explain why you agree or disagree and why.

(i) Student three: *I bet if we try, we can make $\vec{\alpha}$ parallel to torque if the components of $\vec{\alpha}$ are proportional to the elements along the diagonal of the matrix.*

(ii) Student four: *I disagree with student three. But I do think I can make $\vec{\alpha}$ in any direction I want by changing $\vec{\tau}$ and vice versa*

(h) When presented with some new counter-intuitive idea, it is important to make explicit what you find objectionable or worrisome or illogical or contradictory, and try to reconcile it as best you can. What concerns do you have at this time about what we have found? How might you reconcile that with what we know?

(iii) Check with your TA before continuing.

3. What are the eigenvectors and eigenvalues of the inertia tensor a few pages back?

4. Show that if α is parallel to an eigenvector, τ is parallel to α. These eigenvectors are called principle axes.

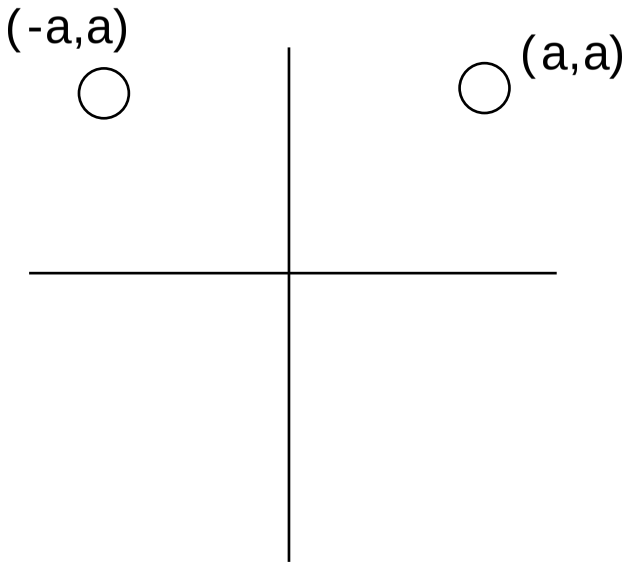
5. Show that if α is not parallel to an eigenvector, τ is not parallel to α. Hint: write α as an linear combination of two eigenvectors, with coefficients a and b.

6. We want to get to the point where we can make a good guess about the principle axes of an object without doing any math. First, convince yourself that if the principle axes coincide with the $x - y - z$ axes, then I will be diagonal.

7. If the tensor is diagonal, it means that all the off-diagonal terms I_{xy} and so on are zero. We need to find out under what conditions that happens. Rank the mass distributions sketched on the last page from highest to lowest values of I_{xy}.

8. State in words what I_{xy} measures.

9. Imagine a scalene a triangle. Estimate the location of the principle axes (with the origin at the center of mass) in the plane of the triangle and give your reasoning.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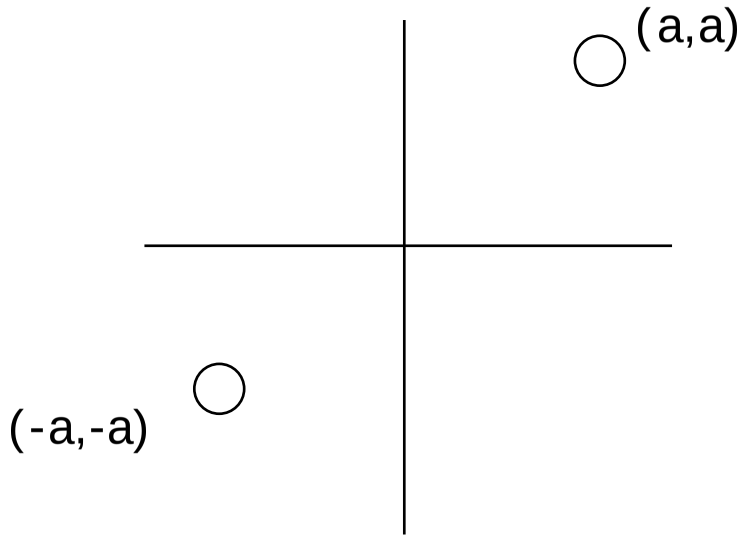
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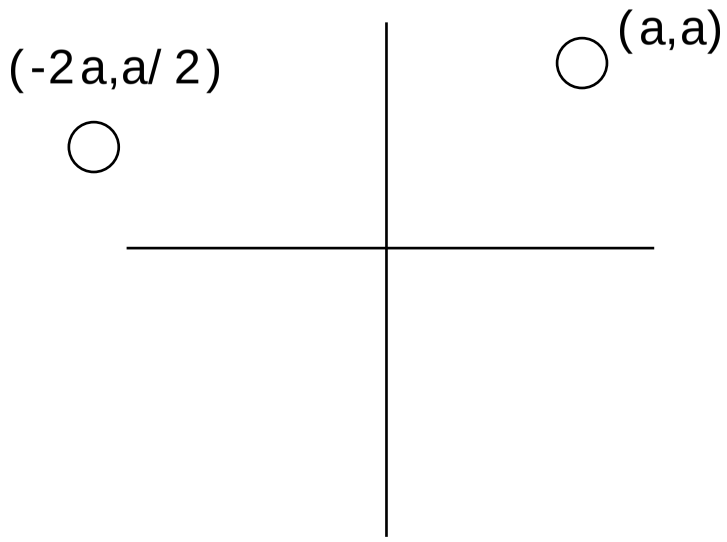
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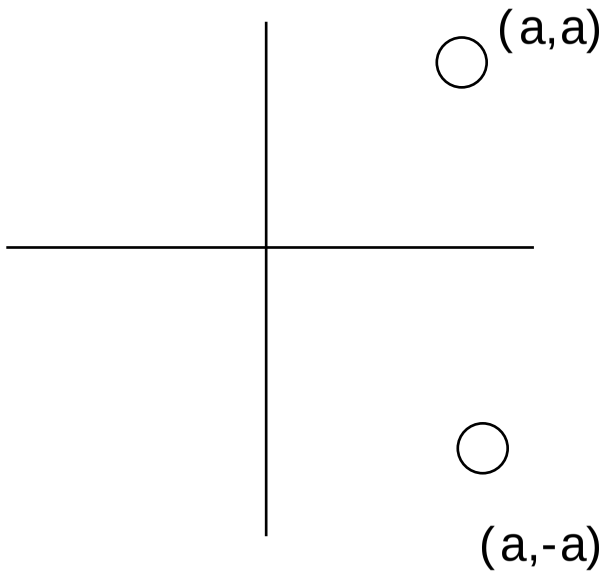
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