Coupled Linear Differential Equations and Phase Space

We will study the phase space trajectories for a damped harmonic oscillator described by the equation:

\[ \ddot{x} + 4\dot{x} + 3x = 0 \]

We will see how the techniques we have learned can give us a global (for all initial conditions) understanding of this system.

1. Using \(\frac{dx}{dt}=v\), change this single equation into two coupled first order equations.

2. Write these equations in matrix form. We will call the constant matrix \(\mathbf{A}\), and the vector \(\mathbf{z}\).

3. We suspect that the solutions will be of the form \(z = Re^{\lambda t}\). However, this cannot be exactly right. Why can’t this be correct? (Hint: there is an important difference between the left and right hand sides of the equation) What is the simplest way to change the guess to make it mathematically correct?
4. A better guess is \( z = Re^nt \). Plug this guess in the differential equation and find the resulting conditions on \( n \) and \( z_0 \).

5. You will have shown that \( n \) and \( z_0 \) are the eigenvalues and eigenvectors of \( A \). Find these eigenvalues and eigenvectors.
6. Because there are two eigenvectors, the general solution is of the form

\[ \vec{z}(t) = R_1 e^{n_1 t} \vec{z}_{01} + R_2 e^{n_2 t} \vec{z}_{02} \]

For simplicity, we will take \( R_1 = R_2 = 1 \) (these values are set by initial conditions). Fill out the table below for various times to see how the relative size of each eigenvector varies with time. (Your table entries should all be numerical values)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( e^{n_1 t} )</th>
<th>( e^{n_2 t} )</th>
<th>( e^{n_1 t} / e^{n_2 t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. As time goes to positive infinity, does your solution tend to one eigenvector or the other? If so, which one? Would your conclusion change if \( 100 R_1 = R_2 \)? If \( R_1 = 100 R_2 \)?

8. As time goes to negative infinity, does your solution tend to one eigenvector or the other? If so, which one? Would your conclusion change if \( 100 R_1 = R_2 \)? If \( R_1 = 100 R_2 \)?
9. On the graph paper given, sketch the eigenvectors (take each grid mark to be one unit both in $x$ and $v$ directions)
10. I have drawn in three possible phase space trajectories (each of them are only partial trajectories). For each trajectory, state if it seems to be obeying the rules of phase space (especially – is the slope equal to acceleration/velocity)?

11. On the same plot, draw at four sample trajectories, each beginning at one of the grey dots. Be sure to use the key properties of phase space trajectories:
   a. Trajectories do not cross
   b. Trajectories are traversed in a clockwise manner
   c. The slope is equal to acceleration/velocity
   d. And, for this case, the relationship of each trajectory to the eigenvectors.

Explain your reasoning for one trajectory.
12. For each of the four trajectories above, sketch a plot of $x(t)$ on the axes below.
(How can you infer from the phase space plot how quickly the particle is moving?)
13. Using phase space information to design a system: Imagine you want to start your system at \( x = 2 \).

   a. What velocity will you give it so that it reaches \( x = 0.2 \) as quickly as possible? Explain your reasoning.

   b. In part a I was looking for an exact value. However, it’s impossible to impart an exact velocity. What ranges of velocity will give you the quickest motion to \( x = 0.2 \)? Explain your reasoning.

   c. Given the above ranges of velocity, what is the maximum time it will take to reach \( x = 0.2 \)?

14. As the damping gets larger and larger (for fixed spring constant), how will the eigenvectors change? Explain your reasoning.