Throughout the year, we will be solving challenging problems by using appropriate approximations. In this sheet, you will review Taylor series as a way to approximate \( \sin(x) \).

1. The general form for the Taylor series is given by

\[
f(x) = \sum_{n=0}^{\infty} (x - x_0)^n \frac{df(x)}{dx} \bigg|_{x_0}
\]

a. What is the \( n=0 \) term if \( x_0=0 \) and \( f(x) = \sin(x) \)?

b. What is the \( n=1 \) term?

c. What is the \( n=2 \) term?

d. What is the \( n=3 \) term?

e. What is the \( n=4 \) term?

f. Based on what you have written, what is the general of the even terms? The odd terms?
2. Let’s think a moment about convergence. The alternating series test states that an alternating series converges if each term gets smaller than the previous term for all \( n > N \) for some integer \( N \), and if the individual terms tend to zero. Given this, for what values of \( x \) is the Taylor series convergent by this test?

3. We can also estimate the error on an alternating series. If the series converges by the alternating series test, then the error in taking only \( N \) terms is less than the absolute value of the \( N+1 \) term (the first term that we ignore). For our series, what is the explicit form of the error if we take only \( N \) terms?

4. Now imagine that you are stuck on an airplane with only a calculator that adds, subtracts, multiplies and divides, but you absolutely must know \( \sin(.5) \) to 1%. Find it now (using only multiplication keys on your calculator) and justify your method.

5. How would you use Taylor series to find \( \sin(\pi/2 + .5) \)? Hint: be careful of convergence issues, but use your knowledge of the sine function. (Do NOT do it, just outline the procedure.)

6. Looking back to your Taylor expansion for \( \sin(x) \) – what must be the units be on \( x \) in order for this series to make sense (in terms of units)? Is your \( x \) in the right units?
Taylor Series in Physics

**Why:** As we solve problems, we will put a lot of emphasis on checking and learning from the solutions. Often we can learn a lot or gain a lot of confidence by looking at limiting cases when the variable gets very big or very small and we have a better grasp of what the solutions should look like from simple arguments. For example, an object subject to drag and gravity with zero initial velocity will have \( v(t) = -gt \) for short times (before the drag force gets large). Taylor series let us do this in a mathematically precise way. Think of the Taylor series as a “zoom tool” that lets us focus in on a small region (typically near zero), or zoom out, and look at the behavior as the variable gets large.

**How:** The general formula for a Taylor series is as follows:

\[
 f(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} \left. \frac{d^n f(x)}{dx^n} \right|_{x=x_0}
\]

We have had some practice with calculating Taylor series using this definition, but there are also tables of Taylor series (see Appendix D in your text) which you are free to use. There are also useful books that have Taylor series, integrals, and so on available at the reference desk in the physics library (Dwight, for example.)

Typically we assume that \( x << 1 \), and so the Taylor series is accurate if we take only a few terms (out to \( n=2 \) or \( 3 \)). If we are looking at large \( x \), then we assume that \( 1/x << 1 \).

The one subtlety that probably did not arise in calculus class is units. The statement that \( x << 1 \) only makes sense if \( x \) is unitless. This requirement also makes sense physically. For example, the statement that a meter is small only makes sense if we compare this to another length (like the size of the earth or the distance to the moon).

Therefore, your first job in doing Taylor expansions is to find a parameter or constant (call it \( a \)) with the same units as your variable (call this \( y \)) and define \( x = y/a \). Something in the problem should cue you that you want \( y << a \) or \( y >> a \). Replace \( y \) with \( xa \) in your formula, and then find the Taylor series in \( x \).

**Example**

We found that

\[
 v(t) = -v_t + (v_t + v_0)e^{-kt}
\]

For a particle subject to both gravity and a linear drag force of magnitude \( kmv \). The terminal velocity is \( g/k \). We expect that if a particle starts from rest \( (v_0=0) \) then for short times, \( v(t) = -gt \) because the drag force is small when the velocity is small.

We will investigate this using the Taylor expansion. First, we have to find a time to compare \( t \) to. By dimensional analysis, we know that \( 1/k \) has units of time. Therefore, we will replace \( t \) by \( x/k \). In the following steps we verified that this gives the correct behavior for \( t << 1/k \) (or \( x << 1 \)).
Taylor Series in Physics

\[ v(x) = -v_t + v_t e^{-x} \quad (v_0 = 0 \text{ for this case}) \]

\[ e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \quad \text{(look up the Taylor expansion)} \]

\[ e^{-x} = 1 - x \quad \text{(for } x \ll 1) \]

(take only the first two terms since we expect an answer linear in \( t \))

\[ v(x) = -v_t + v_t (1 - x), \quad \text{(plug back in)} \]

\[ v(x) = -v_t x \quad \text{(simplify)} \]

\[ v(t) = -\frac{g}{k} \star kt = -gt \quad \text{(plug back in original parameters and variables)} \]

Which is exactly what we hoped to find.

**Practice** We will practice with an example from electrostatics: Imagine that you and your colleague have just independently calculated the electric field due to a charged disk with radius \( R \) and surface charge density \( \sigma \). You are calculating the field at a point a distance \( z \) away from the center of the disk, and directly above it. You have different answers (\( E_1 \) and \( E_2 \)) and need to decide which is correct.

\[ E_1 = \frac{\sigma}{2\varepsilon_0} \left( 1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \quad E_2 = \frac{\sigma}{2\varepsilon_0} \left( 1 - \frac{z^2}{z^2 + R^2} \right) \]

1. Use dimensional analysis to check both of these. Do both pass this test? (Recall that \( E = q/(4\pi\varepsilon_0 r^2) \) for a point charge).
2. If \( z \) is much less than \( R \), the disk will look like an infinite sheet of charge. In this case, the electric field should be \( E = \sigma/(2\varepsilon_0) \). Do both pass this test? (Here you can say that \( z=0 \)).
3. If you get far enough away from the disk, the field should go to zero. Do both pass this test? (Here you can say \( R=0 \)).
4. A more stringent test is that if you get far enough away from the disk, the field should be exactly the same as if it were a point particle (\( E = q/(4\pi\varepsilon_0 r^2) \)). Do both pass this test? Hint: what is \( \sigma \) in terms of \( q \) and \( R \)? Here you need to use Taylor expansions, with \( z \gg R \).